MINIMIZING THE MODULUS OF THE REFLECTION COEFFICIENT OF A POLYCHROMATIC WAVE FROM AN INHOMOGENEOUS ABSORBING LAYER

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The synthesis is considered of the optimum inhomogeneous absorbing layer of given thickness under the incidence of a plane polychromatic wave with a known frequency spectrum. The synthesis is treated as a problem in optimum control. The control is approximated as a step function and the problem is reduced to minimizing a function of many variables. Equations are obtained for the exact calculation of the gradient in the goodness criterion; this gradient is required in setting up the algorithm for finding the optimum solution. The principal and adjoint variables and also the Hamiltonian are written in complex form and this greatly simplifies the intermediate transformations. The optimum solution is sought by using the method of adjoint gradients [1, 2] with constraints placed on the control. To illustrate the problem, results are given of computer calculations of an optimum control.

1. Statement of the Problem

Suppose that a plane polychromatic wave of known frequency spectrum $\varkappa_j = f_j/f_1$ (f_1 is the lowest frequency in the spectrum; $f_2, f_3, \ldots, f_s$ are the remaining frequencies) is incident normally on an inhomogeneous absorbing layer which is separated from a homogeneous half-space by a boundary which is characterized by an arbitrary complex admittance.

The system of differential equations for the dimensionless input admittances G_j of an inhomogeneous layer has the form [3]

$$dG_j/d\tau = -i\varkappa_j(n_0^2 - G_j^2), \qquad j = 1, \ldots, s; \qquad n_0^2 = 1 + (1 + i\eta)Q(\tau), \tag{1.1}$$

where G_j is the input admittance of the layer relative to the characteristic admittance $(1/\rho_0 c_0)$ of the medium from which the wave is coming; ρ_0, c_0 are the density of the medium and the velocity of a longitudinal wave in the medium; $n_0 = c_0/c(\tau)$, $c(\tau)$ is the velocity in the inhomogeneous absorbing layer; $\tau = (2\pi f_1/c_0)x$ is the reduced thickness of the layer bounded by the planes $x = 0$ and $x = l$; x is the coordinate; η is a given positive constant; $Q(\tau)$ is a nonnegative function which satisfies the condition

$$0 \leqslant Q(\tau) \leqslant M. \tag{1.2}$$

In accordance with the terminology used in optimal control theory we henceforward refer to $Q(\tau)$ as the control function.

We take the values of the input admittances on the boundary $\tau = 0$ to be

$$G_j(0) = p_0 + iq_0, \quad j = 1, \ldots, s. \tag{1.3}$$

The reflection coefficient of a monochromatic sound wave depends on the frequency and is determined by the value of the dimensionless input admittance $G_j = p_j + iq_j$; on the boundary $\tau = \tau_l$ [4]

$$\beta_j = (G_j^{(l)} - 1)/(G_j^{(l)} + 1), \qquad j = 1, \ldots, s.$$

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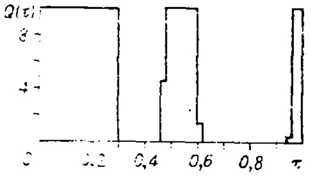

Fig. 1

Fig. 2

For a polychromatic wave we can characterize the reflection by means of the energy coefficient

$$\gamma = \sum_{j=1}^{s} e_j |\beta_j|^2 = \sum_{j=1}^{s} e_j \frac{(p_j^{(l)} - 1)^2 + (q_j^{(l)})^2}{(p_j^{(l)} + 1)^2 + (q_j^{(l)})^2}, \tag{1.4}$$

where e_j is the energy spectrum of the wave and p_j and q_j are the real and imaginary parts of the acoustic admittance G_j.

Thus the problem of optimum synthesis consists in finding the control function $Q(\tau)$ which satisfies condition (1.2) and the minimizing functional (1.4).

2. The Construction of an Approximate Model of the Optimum Synthesis Problem

As was suggested in [5] we make use of the auxiliary complex form for writing the adjoint variables (Lagrange functions) $\lambda_j^* = \lambda_p^j - i\lambda_q^j$. We can write in the same form the Hamiltonian

$$\overline{H}_\lambda = \sum_{j=1}^{s} \lambda_j^* \frac{dG_j}{d\tau} = -i \sum_{j=1}^{s} \varkappa_j \lambda_j^* \left[1 + (1 + i\eta) Q(\tau) - G_j^2\right]. \tag{2.1}$$

whose real part is used to construct the Pontryagin maximum principle in the theory of optimum systems. Since the Hamiltonian (2.1) depends linearly on the control $Q(\tau)$, we can assume in accordance with this principle that the optimum controls are to be found among piecewise-constant functions and can take only extremal values; i.e., the optimum inhomogeneous absorbing layer can consist of alternating homogeneous controls with zero and maximum values. The number of homogeneous layers which make up the optimum inhomogeneous layer is determined by the number of zeros in the switching function

$$K = \mathrm{Re}\, \frac{\partial \overline{H}_\lambda}{\partial Q} = -\mathrm{Re}\left[i \sum_{j=1}^{s} (1 + i\eta) \varkappa_j \lambda_j^*\right].$$

For positive values of the switching function the control is equal to zero, and for negative values it is a maximum. The optimum synthesis problem therefore reduces to finding the zeros of the switching function and this involves solving a two-point boundary-value problem.

It is, however, possible to approach the problem in another way, starting from an approximate model. We assume that the inhomogeneous absorbing layer consists of n homogeneous layers of equal thickness, for each of which the control function is constant and equal to Q_k $(k = 1, \ldots, n)$, $0 \leq Q_k \leq M$. The differential equations for the input admittances (1.1) for each of the layers can now be integrated:

$$G_j^k (h - 0) = \frac{G_j^k (+0) - i\delta_k \operatorname{tg} \psi_j^k}{1 - iG_j^k (+0)\, \delta_k^{-1} \operatorname{tg} \psi_j^k}, \tag{2.2}$$

$$h = \tau_l / n, \qquad \psi_j^k = h \varkappa_j \delta_k, \qquad \delta_k = \sqrt{1 + Q_k + i\eta Q_k},$$
$$k = 1, \ldots, n, \qquad j = 1, \ldots, s.$$

Since the acoustic admittance changes discontinuously through the boundary between the two media, i.e.,

$$G_j^{k+1} (+0) = G_j^k (h - 0),$$

the system (2.2) can be written in the recurrent form

$$G_j (k) = \frac{G_j (k-1) - i\delta_k \operatorname{tg} \psi_j^k}{1 - iG_j (k-1)\, \delta_k^{-1} \operatorname{tg} \psi_j^k}, \tag{2.3}$$

$$h = \tau_l / n, \qquad \psi_j^k = h \varkappa_j \delta_k, \qquad \delta_k = \sqrt{1 + Q_k + i\eta Q_k},$$
$$k = 1, \ldots, n, \qquad j = 1, \ldots, s.$$

and when the control vector $\{Q_k\}$ and the initial conditions (1.3) are known we can uniquely determine the value of the admittance $G_j^{(l)} = G_j(n)$ on the boundary with the external medium and thus calculate the reflection coefficient (1.4).

Thus, the reflection coefficient of a polychromatic wave is a unique function of the n variables $Q_k(k = 1, \ldots, n)$ which satisfy condition (1.2), and the problem of optimal control reduces to the minimization of a function of many variables.

We might expect from physical considerations that the approximate model of an optimum inhomogeneous absorbing layer will be sufficiently accurate providing the thickness of the homogeneous layers $h = \tau_l/n$ is smaller than the thickness of a homogeneous layer with $Q_n = M$ which would pass a significant amount of the energy of an incident wave at the highest frequency of the linear spectrum.

3. Calculation of the Gradient of the Minimizing Function

Effective numerical methods of solving optimization problems require a knowledge of the gradient of the minimizing function or a method for calculating this gradient. In the present problem with an arbitrary number of variables n, the minimizing function cannot be written analytically; however, its gradient can be calculated exactly by the method described in [6].

As we pointed out, we make use of the complex form of the relevant expressions in order to simplify the intermediate transformations which occur in the derivation of the equations for the gradient of the goodness criterion. Thus, in the formulated discrete problem (2.3), (1.3), (1.2), and (1.4) of the optimum control we can use the expanded criterion

$$J = \gamma - \sum_{k=1}^{n} \mathrm{Re} \sum_{j=1}^{s} \lambda_j^*(k) [\varphi(G_j(k-1), Q_k, \varkappa_j) - G_j(k)]. \tag{3.1}$$

where $\varphi(G_j(k-1), Q_k, \varkappa_j)$ are the right sides of (2.3); $\lambda_j^*(k) = \lambda_p^j(k) - i\lambda_q^j(k)$ are the complex Lagrange multipliers. It can be seen that the expanded criterion (3.1) is identical with the functional (1.4) when condition (2.3) is satisfied.

We introduce the complex-function series

$$\overline{H}_\lambda(k) = \sum_{j=1}^{s} \lambda_j^*(k) \varphi(G_j(k-1), Q_k, \varkappa_j), \qquad k = 1, \ldots, n, \tag{3.2}$$

so that the first variation of the functional (3.1)

$$\delta J = \mathrm{Re} \sum_{j=1}^{s} \left\{ \left[\frac{\partial\gamma}{\partial p_j(n)} - i\frac{\partial\gamma}{\partial q_j(n)}\right] \delta G_j(n) - \lambda_j^*(n)\,\delta G_j(n) + \sum_{k=1}^{n} \left[\frac{\partial \overline{H}_\lambda(k)}{\partial G_j(k-1)} - \lambda_j^*(k-1)\right] \delta G_j(k-1) \right\} + \mathrm{Re} \sum_{k=1}^{n} \frac{\partial \overline{H}_\lambda(k)}{\partial Q_k}\,\delta Q_k$$

can be obtained in the form

$$\delta J = \mathrm{Re} \sum_{k=1}^{n} \frac{\partial \overline{H}_\lambda(k)}{\partial Q_k}\,\delta Q_k, \tag{3.3}$$

if the complex Lagrange multipliers are chosen to satisfy the conditions

$$\lambda_j^*(n) = \partial\gamma/\partial p_j(n) - i\partial\gamma/\partial q_j(n), \qquad j = 1, \ldots, s;$$
$$\lambda_j^*(k-1) = \partial \overline{H}_\lambda(k)/\partial G_j(k-1), \qquad j = 1, \ldots, s, \qquad k = n, \ldots, 1. \tag{3.4}$$

From (3.2) and (2.3) this condition becomes

$$\lambda_j^*(k-1) = [\lambda_j^*(k)(1 + \mathrm{tg}^2 \psi_j^k)]\,[1 + iG_j(k)\,\delta_k^{-1}\,\mathrm{tg}\,\psi_j^k]^{-2},$$
$$j = 1, \ldots, s, \qquad k = n, \ldots, 1. \tag{3.5}$$

From (3.3) for the variation of the functional we get the equations for the components of the gradient of the goodness criterion

$$\partial J/\partial Q_k = \mathrm{Re}\,\partial \overline{H}_\lambda(k)/\partial Q_k, \quad k = 1, \ldots, n. \tag{3.6}$$

Replacing the complex functions $\bar{H}_\lambda(k)$ in (3.6) by their expressions in terms of the principal variables $G_j(k-1)$ and the Lagrange multipliers $\lambda_j^*(k)$, we obtain the final equations for calculating the components of the gradient of the criterion from the known solutions of the straight lines (1.3) and (2.3) and of the discrete equations inverse to (3.4) and (3.5):

$$\frac{\partial J}{\partial Q_k} = \operatorname{Re} \sum_{j=1}^{s} \frac{(n-i)}{2\delta_k} \left\{ \left[\lambda_j^*(k-1) + \frac{G_j^2(k)}{\delta_k^2} \lambda_j^*(k) \right] \operatorname{tg} \psi_j^k + \left[1 - \frac{G_j^2(k)}{\delta_k^2} \right] \psi_j^k \lambda_j^*(k) \right\},$$
$$k = 1, \ldots, n.$$

4. Minimization Algorithm and Results of the Calculations

We have considered the minimization of the functional (1.4) for the case where the energy of the incident wave is distributed uniformly over the different frequencies, i.e., where $e_j = 1/s$. A preliminary analysis of the reflection coefficient (1.4) as a function of the control parameters Q_k ($k = 1, \ldots, n$) showed that there were deep chasms in the functional which make it difficult to find the optimum solution. Functionals with this type of "relief" are characteristic of synthesis problems in both acoustic and optical multilayer media [7].

In order to find the optimum solution we have used the two-step procedure of the adjoint gradient method [1, 2], since the simplest one-step algorithm of steepest descent shows only slow convergence when there are chasms in the minimizing function. The chosen algorithm is

$$Q^{m+1} = Q^m + \alpha_m p^m;$$

$$p_k^m = \begin{cases} -\dfrac{\partial J(Q^m)}{\partial Q_k} + \beta_m p_k^{m-1}, & k \notin I_m, \\ 0, & k \in I_m; \end{cases}$$

$$\alpha_m: \quad J(Q^m + \alpha_m p^m) = \min_{0 \leqslant \alpha \leqslant \bar{\alpha}_m} J(Q^m + \alpha p^m),$$

$$\bar{\alpha}_m = \max\{\alpha: \quad 0 \leqslant Q_k^m + \alpha p_k^m \leqslant M, \quad k = 1, \ldots, n\};$$

$$\beta_m = \begin{cases} \dfrac{\sum_{k \in \bar{I}_m} \left(\partial J(Q^m)/\partial Q_k\right)^2}{\sum_{k \in \bar{I}_m} \left(\partial J(Q^{m-1})/\partial Q_k\right)^2}, & k \neq 0, \quad I_m = I_{m-1}, \\ 0, & k = 0 \text{ or } I_m \neq I_{m-1}, \end{cases}$$

$$I_m = I_m^- \cup I_m^+;$$

$$I_m^- = \begin{cases} \{k: \quad Q_k^m = 0, \quad \partial J(Q^m)/\partial Q_k > 0\}, & m = 0 \text{ or} \\ \partial J(Q^m)/\partial Q_k = 0 \quad \text{for all} & k \notin I_m, \\ I_{m-1}^- \cup \{k: Q_k^m = 0\} & \text{in the remaining cases;} \end{cases}$$

$$I_m^+ = \begin{cases} \{k: \quad Q_k^m = M, \quad \partial J(Q^m)/\partial Q_k < 0\}, & m = 0 \text{ or} \\ \partial J(Q^m)/\partial Q_k = 0 \quad \text{for all} & k \notin I_m, \\ I_{m-1}^- \cup \{k: Q_k^m = M\} & \text{in the remaining cases.} \end{cases}$$

By means of this algorithm we have synthesized an optimum layer with a reduced thickness $\tau_l = 1$ and an absorption coefficient $\eta = 0.7$. The control function which characterizes the properties of the material was taken to be in the region $0 \leq Q(\tau) \leq 10$. It was assumed that the absorbing layer was bounded by a medium with zero admittance ($p_0 = 0$, $q_0 = 0$) and was acted on by a plane polychromatic wave with the frequency spectrum $\varkappa_1 = 1$, $\varkappa_2 = 1.5$, $\varkappa_3 = 2$, $\varkappa_4 = 2.5$.

In order to carry out the iteration procedure for finding the optimum inhomogeneous layer we used an initial control corresponding to the optimum homogeneous layer. A large number of control parameters ($n = 50$) was chosen, and this enabled us to find a solution (Fig. 1) which was close to the optimum solution for the continuous problem. The energy reflection coefficient was found as $\gamma_{min} = 0.080056825$. The result confirms the relay nature of the optimum control which we mentioned in §2.

If only a small number ($n = 5$) of parameters is used, it is not possible to see the on/off nature of the optimum control (Fig. 2). However, such an approach has the advantage that it requires a shorter calculation time, and it gives a satisfactory value of the minimal reflection coefficient ($\gamma_{min} = 0.096415685$).

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DYNAMICS OF THIN FILMS IN A MAGNETIC FIELD

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The oscillations of thin conducting films placed in a magnetic field are considered. The effect of the field in different directions on the effective elasticity of the film is described and dispersion relations are obtained for longitudinal and transverse waves.

It is well known from the theory of elasticity [1] that the properties of strain waves in an isotropic medium are different from those of waves in thin films. For example, waves which are normal to the plane of the film exhibit dispersion and the phase velocity of longitudinal waves is altered. We might expect that new deformation modes will occur in a conducting film placed in a magnetic field and that the properties of these will differ from those of magnetic-field transport waves owing to the deformations in the three-dimensional elastic conducting medium.

Effects related to the presence of an external magnetic field should begin to appear at much smaller field values because the characteristic velocity in a magnetic field increases as the thickness of the film is reduced. For thin conducting films it is possible by proper choice of the parameters to get the magnetoelastic velocity greater than the velocity of sound; i.e., the nature of the strain propagation in the film will be determined mainly by the magnetic field.

We shall consider the propagation of deformations in a thin perfectly conducting film of thickness d placed in an external homogeneous constant magnetic field **H**. In order to get the equation for the deformation **u**, we make use of the equilibrium equations for a thin elastic film [1]. We introduce a Cartesian system of coordinates x, y, z so that the film lies in the (x, y) plane and the external magnetic field is in the (x, z) plane at an angle α to the x axis. For the displacement u_z we have

$$[Ed^3/12(1-\tau^2)]\,\Delta^2 u_z - P_z = 0. \tag{1}$$

For the displacements u_x and u_y we have

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